Models in Epidemiology

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The Black Death

The black death was one of the deadliest epidemics in human history. It killed tens of millions in Europe around 1350.

"You must picture the consternation of our little town, hitherto so tranquil, and now, out of the blue, shaken to its core, like a quite healthy man who all of a sudden feels his temperature shoot up and the blood seething like wildfire in his veins."

– Albert Camus, “The Plague”

THE SIR MODEL

● The progress of individuals is schematically described by:
$S \rightarrow I \rightarrow R$.

● The model is given by:

(1) $S + I \rightarrow I + I$ (with rate constant r)

(2) $I \rightarrow R$ (with rate constant a).

● The SIR system described above can be expressed by the following set of ordinary differential equations:

$$(3) \quad \frac{dS}{dt} = -rSI$$

$$(4) \quad \frac{dI}{dt} = rSI - aI$$

$$(5) \quad \frac{dR}{dt} = aI$$

- $(3),(4),(5) \Rightarrow \frac{dS}{dt}+\frac{dI}{dt}+\frac{dR}{dt}=0 \Rightarrow S(t)+I(t)+R(t)=N$
- $(3) \Rightarrow \frac{dS}{dt}<0 \Rightarrow S(t)\downarrow \Rightarrow S(t)\le S_0, t\ge 0$
- $S_0<\frac{a}{r}=\rho \Leftrightarrow rS_0-a<0; (4) \Rightarrow \frac{dI}{dt}=I(rS-a)<0 \Rightarrow 0\le \lim_{t\to\infty} I(t)<\varepsilon.$
- $S_0>\frac{a}{r}=\rho \Leftrightarrow rS_0-a>0; (4) \Rightarrow \frac{dI}{dt}=I(rS-a)>0 \Rightarrow I(t)\uparrow.$
- $(3),(4) \Rightarrow \frac{dI}{dS}=\frac{\frac{dI}{dt}}{\frac{dS}{dt}}=\frac{(rS-a)I}{-rSI}=-1+\frac{a}{rS}=-1+\frac{\rho}{S}$

$$\Rightarrow dI=(-1+\frac{\rho}{S})dS \Rightarrow I=-S+\rho\ln(S)+C$$

$$\Rightarrow I+S-\rho\ln(S)=I_0+S_0-\rho\ln(S_0) (*)$$

- $\frac{dI}{dt} = 0 \Rightarrow (rS - a)I = 0 \Rightarrow rS - a = 0 \Rightarrow S = \frac{a}{r} = \rho$

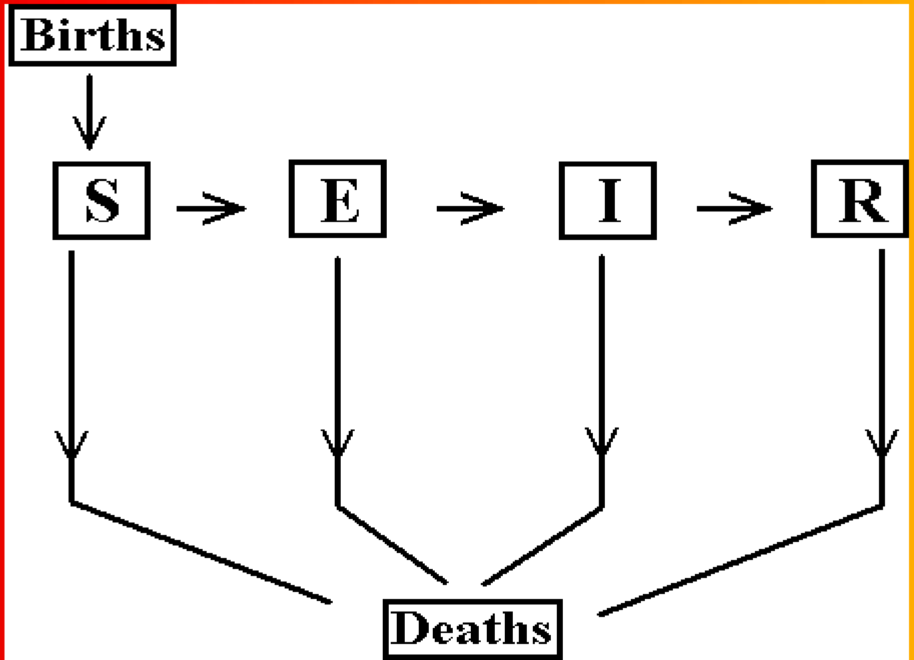
$$(*) \Rightarrow I_{\max} = I_0 + S_0 - \rho \ln(S_0) - S + \rho \ln(S) =$$

$$I_0 + S_0 - \rho \ln(S_0) - \rho + \rho \ln(\rho) = (I_0 + S_0) - \rho + \rho[\ln(\rho) - \ln(S_0)] =$$

$$N - \rho + \rho \ln(\frac{\rho}{S_0})$$

THE SEIR MODEL

- We now turn our attention to the article written by Schaffer and Bronnikova. This article explores how arbitrarily small changes in parameter values can induce qualitative changes in behavior.
- A simple modification of the SIR Model is used which incorporates an exposed class.
- We will also assume that individuals enter the S class at birth and exit the S, E, I, and R classes at death.



THE MODEL AFTER NORMALIZATION

- $s = \frac{S}{N}; e = \frac{E}{N}; i = \frac{I}{N}; \beta_0 = B_0 N$
- $\frac{ds}{dt} = m(1 - s) - \beta(t) si$

$$\frac{de}{dt} = \beta(t) si - (m + a) e$$

$$\frac{di}{dt} = ae - (m + g) i$$

$$\beta(t) = \beta_0 [1 + \varepsilon_\beta \cos(2\pi t)]$$

IN THE ABSENCE OF SEASONALITY

- $no - disease \quad state:$

$$(s^0, e^0, i^0) = (1,0,0)$$

- $endemic \quad state:$

$$(s^*, e^*, i^*) = \left(\frac{1}{R_0}, \frac{m}{m+a}\left(1 - \frac{1}{R_0}\right), \frac{am}{(m+g)(m+a)}\left(1 - \frac{1}{R_0}\right)\right)$$

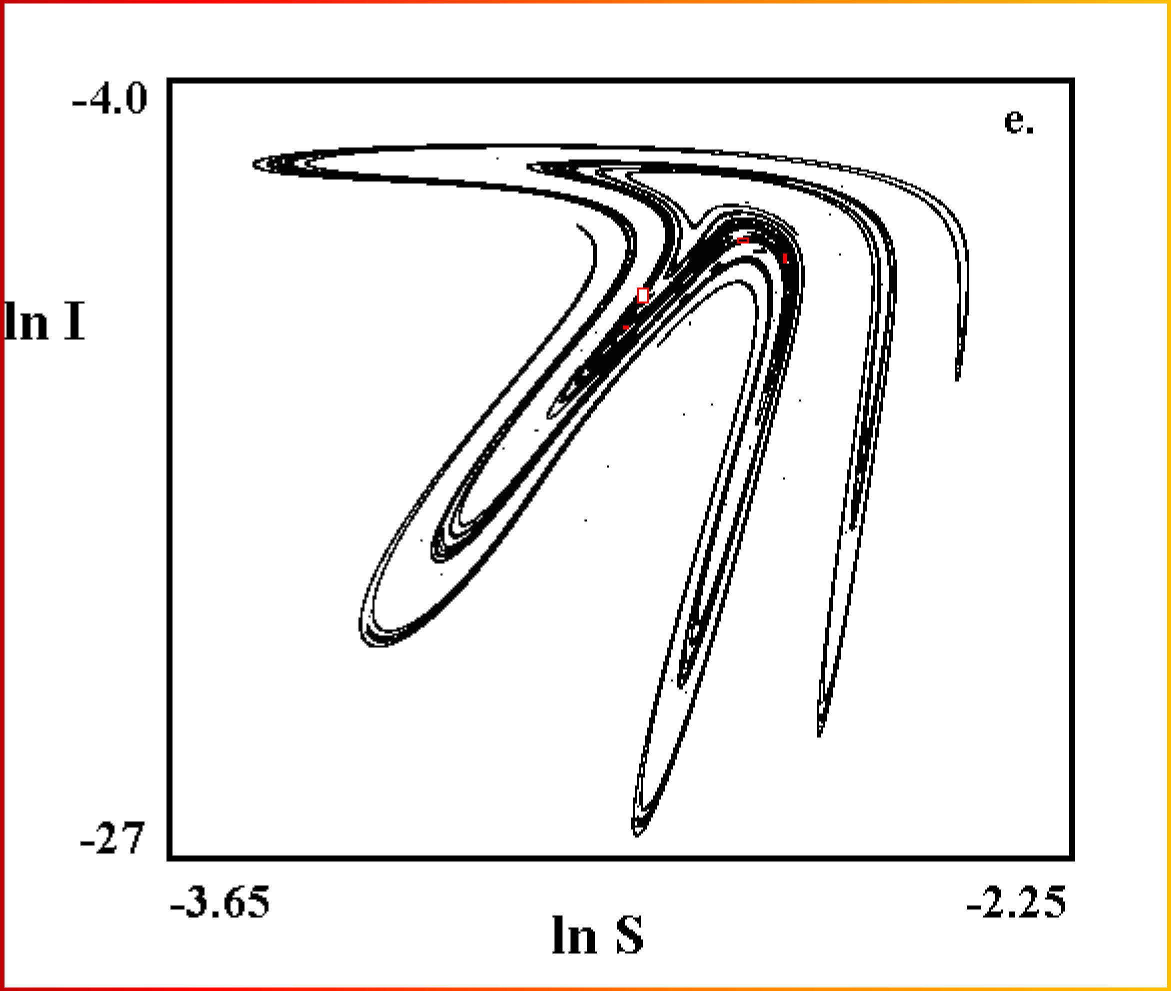
$$where \quad R_0 = \frac{a\beta_0}{(m+a)(m+g)}$$

- If $R_0<1$ then the no disease equilibrium is stable and the disease dies out.
- If $R_0>1$ then the endemic equilibrium is stable and the disease persists.

IN THE PRESENCE OF SEASONALITY

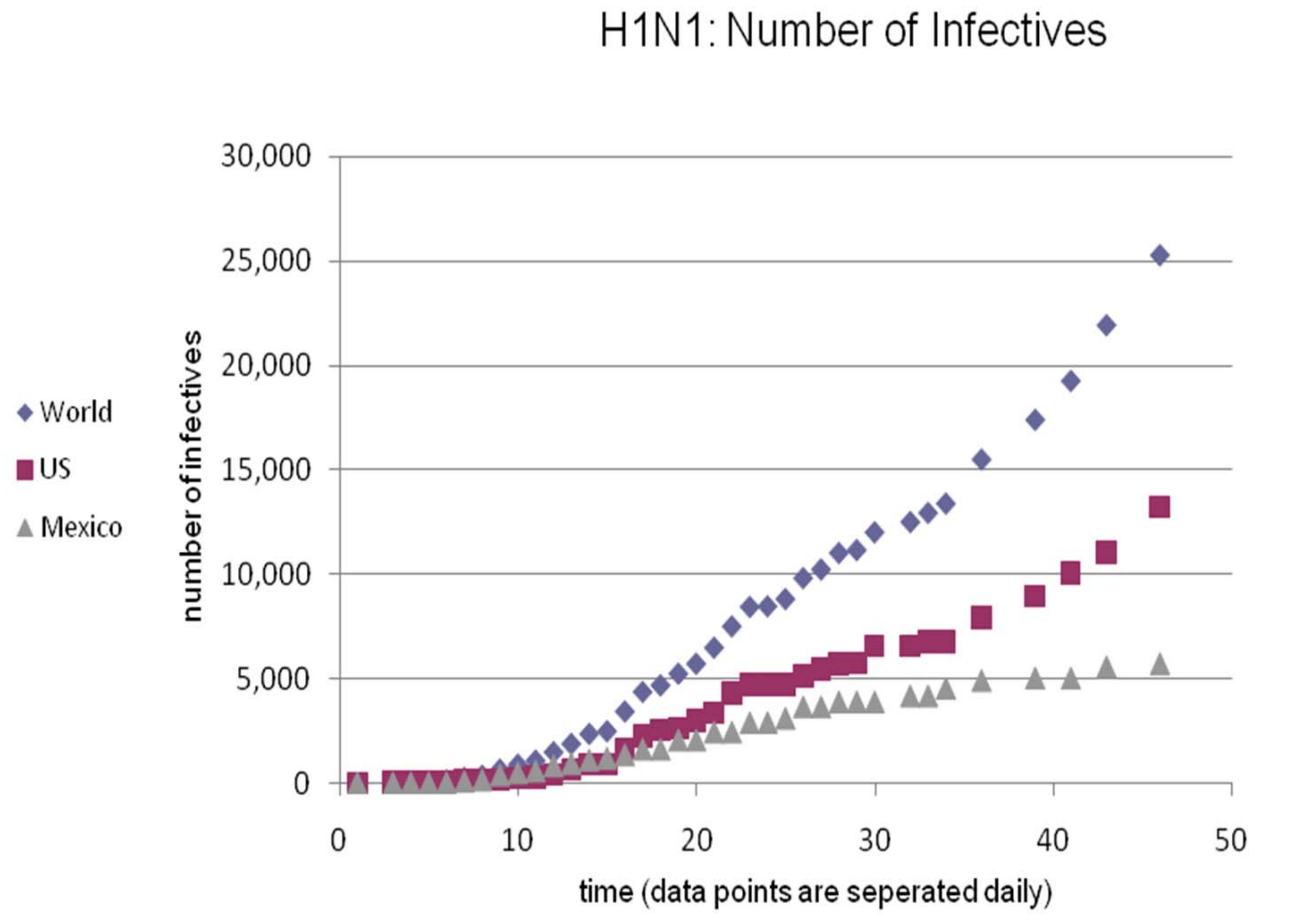
- With the addition of seasonality a wide range of dynamics is manifested:
- Main period-doubling sequence: these motions bear the imprints of seasonality.
- Coexisting subharmonic resonances: the mark of seasonality is absent.
- For certain parametric values we observe chaotic behavior.
- Chaos manifests itself as an exponential growth of initial conditions, thus giving the impression that the system is random although it is in fact deterministic.
- Chaotic systems exhibit strong unpredictability which is not shown by other deterministic systems.

NON-STABLE CHAOS



BBSI RESEARCH

- Swine influenza virus is hosted (and is endemic) in pigs.
- The 2009 swine flu outbreak was due a new strain of the influenza A virus classified as subtype H1N1.
- This strain can pass from human to human and causes normal symptoms of influenza.
- The virus binds to an epithelial cells in the lung and throat through interactions between hemagglutinin and the cell surface.
- The genome contains eight pieces of segmented RNA which encodes eleven proteins.
- RNA proofreading enzymes are absent; eight separate segments of RNA allows mixing → high mutation rate.



H1N1: Number of Infectives
30,000
25,000
20,000
15,000
10,000
5,000
0
number of infectives
0
10
20
30
40
50
time (data points are seperated daily)
World
US
Mexico